Chapter 2

Topology of Kähler manifolds

2.1 Chern class

Definition 2.1.1. Let M be a smooth closed manifold with dimensional m. For $p \in \mathbb{Z}$, $0 \le p \le m$, the p-th **de Rham cohomology** of M (with complex coefficient) is defined by

$$H^p_{\mathrm{dR}}(M, \mathbb{C}) = \ker d|_{\Omega^p(M)} / d\Omega^{p-1}(M). \tag{2.1.1}$$

The (total) **de Rham cohomology** of M is defined as

$$H^*_{\mathrm{dR}}(M, \mathbb{C}) = \bigoplus_{p=0}^{m} H^p_{\mathrm{dR}}(M, \mathbb{C}). \tag{2.1.2}$$

From Definition 2.1.1, we see that any closed differential form ω on M, i.e., $d\omega = 0$, determines a cohomology class $[\omega] \in H^*_{\mathrm{dR}}(M, \mathbb{C})$. Moreover, two closed differential forms ω, ω' on M determine the same cohomology class if and only if there exists $\eta \in \Omega^*(M)$ such that $\omega - \omega' = d\eta$.

If ω, ω' are two closed differential forms on M and a is a constant function on M, then

$$[a\omega] = a[\omega], \quad [\omega + \omega'] = [\omega] + [\omega'] = [\omega'] + [\omega]. \tag{2.1.3}$$

Moreover, for $\eta, \eta' \in \Omega^*(M)$,

$$(\omega + d\eta) \wedge (\omega' + d\eta') = \omega \wedge \omega' + d(\eta \wedge \omega' + (-1)^{\deg \omega} \omega \wedge \eta' + \eta \wedge d\eta'). \tag{2.1.4}$$

Thus the cohomology class $[\omega \wedge \omega']$ depends only on $[\omega]$ and $[\omega']$. We denote it by $[\omega] \cdot [\omega']$. If ω'' is another closed differential forms on M, then

$$([\omega] + [\omega']) \cdot [\omega''] = [\omega] \cdot [\omega''] + [\omega'] \cdot [\omega'']. \tag{2.1.5}$$

From the above discussion, the de Rham cohomology of M carries a natural ring structure.

The importance of the de Rham cohomology lies in the de Rham theorem as follows.

Theorem 2.1.2. *Let M be a smooth closed oriented manifold with dimensional m. For $p \in \mathbb{Z}$, $0 \leq p \leq m$,*

(1) $\dim H^p_{\mathrm{dR}}(M, \mathbb{C}) < +\infty$;

(2) $H^p_{\mathrm{dR}}(M, \mathbb{C})$ is canonically isomorphic to $H^p_{\mathrm{Sing}}(M, \mathbb{C})$, the p-th singular cohomology of M.

By Theorem 2.1.2, we could see that $H^*_{\mathrm{dR}}(M, \mathbb{C})$ is a topological invariant, although we construct it from the differential structure and the differential forms. We usually simply denote it by $H^*(M, \mathbb{C})$.

Let E be a complex vector bundle over M. Recall that in (1.2.9), we interpret the curvature $R^E \in \Omega^2(M, \mathrm{End}(E))$ as the composition of connections. Furthermore, in view of the composition of the endomorphisms, for any $k \in \mathbb{N}$,

$$\left(R^E\right)^k = \overbrace{R^E \circ \cdots \circ R^E}^{k} : \mathscr{C}^\infty(M, E) \to \Omega^{2k}(M, E) \tag{2.1.6}$$

is a well-defined element lying in $\Omega^{2k}(M, \mathrm{End}(E))$.

For any $A \in \mathscr{C}^\infty(M, \mathrm{End}(E))$, the fiberwise trace of A forms a smooth function on M. We denote this function by $\mathrm{tr}[A]$. This further induces the map

$$\mathrm{tr} : \Omega^*(M, \mathrm{End}(E)) \to \Omega^*(M) \tag{2.1.7}$$

such that for any $\omega \in \Omega^*(M)$ and $A \in \Omega^*(M, \mathrm{End}(E))$,

$$\mathrm{tr}(\omega A) = \omega\, \mathrm{tr}[A]. \tag{2.1.8}$$

We also extend the Lie bracket operation on $\mathrm{End}(E)$ to $\Omega^*(M, \mathrm{End}(E))$ as follows: for any $\omega, \eta \in \Omega^*(M)$ and $A, B \in \Omega^*(M, \mathrm{End}(E))$,

$$[\omega A, \eta B] = (\omega A)(\eta B) - (-1)^{\deg \omega \cdot \deg \eta} (\eta B)(\omega A). \tag{2.1.9}$$

The following Proposition is obvious by (2.1.9).

Proposition 2.1.3. *For any $A, B \in \Omega^*(M, \mathrm{End}(E))$,*

$$\mathrm{tr}\left[[A, B]\right] = 0. \tag{2.1.10}$$

Proposition 2.1.4. *Let ∇^E be a connection on E. Then for any $A \in \Omega^*(M, \mathrm{End}(E))$,*

$$d\,\mathrm{tr}[A] = \mathrm{tr}\left[[\nabla^E, A]\right], \tag{2.1.11}$$

where

$$[\nabla^E, A] = \nabla^E \circ A - (-1)^{\deg A} A \circ \nabla^E \tag{2.1.12}$$

as in (2.1.9).

Proof. First of all, if $\widetilde{\nabla}^E$ is another connection on E, then by (1.2.3), $\nabla^E - \widetilde{\nabla}^E \in \Omega^1(M, \mathrm{End}(E))$. Thus by (2.1.3), we have $\mathrm{tr}\left[[\nabla^E - \widetilde{\nabla}^E, A]\right] = 0$. So the right hand side of (2.1.11) does not depend on the choice of ∇^E.

On the other hand, by (2.1.12), the right hand side of (2.1.11) is local. Thus for any $x \in M$, we could choose a sufficiently small open neighbourhood U_x of x such that $E|_{U_x}$ is trivial. Then we can take a trivial connection on $E|_{U_x}$ for which (2.1.11) holds obviously.

By combining the above independence and local properties, (2.1.11) holds on the whole manifold M.

The proof of our proposition is completed. □

Let

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots, \quad a_i \in \mathbb{C}, \tag{2.1.13}$$

be a power series in one variable. Since $R^E \in \Omega^2(M, \mathrm{End}(E))$,

$$\mathrm{tr}\left[f(R^E)\right] = a_0 + a_1\,\mathrm{tr}[R^E] + \cdots + a_n\,\mathrm{tr}\left[(R^E)^n\right] + \cdots \tag{2.1.14}$$

is an element in $\Omega^*(M, \mathbb{C})$, which only have finite terms.

We now state a form of the Chern-Weil theorem as follows.

Theorem 2.1.5. *(1) The form $\mathrm{tr}\left[f(R^E)\right]$ is closed. That is,*

$$d\,\mathrm{tr}\left[f(R^E)\right] = 0. \tag{2.1.15}$$

(2) If $\widetilde{\nabla}^E$ is another connection on E with curvature $\widetilde{R}^E$, then there is a differential form $\omega \in \Omega^*(M, \mathbb{C})$ such that

$$\mathrm{tr}\left[f(R^E)\right] - \mathrm{tr}\left[f(\widetilde{R}^E)\right] = d\omega. \tag{2.1.16}$$

Proof. (1) From Proposition 2.1.4,

$$d\,\mathrm{tr}\left[f(R^E)\right] = \mathrm{tr}\left[[\nabla^E, f(R^E)]\right] = \sum_i \mathrm{tr}\left[a_i\left[\nabla^E, (R^E)^i\right]\right] = 0 \tag{2.1.17}$$

as we have the Bianchi Identity (cf. Proposition 1.2.7)

$$\left[\nabla^E, (R^E)^i\right] = \left[\nabla^E, (\nabla^E)^{2i}\right] = 0. \tag{2.1.18}$$

(2) For any $t \in [0,1]$, let ∇_t^E be the deformed connection on E given by

$$\nabla_t^E = (1-t)\nabla^E + t\widetilde{\nabla}^E. \tag{2.1.19}$$

Then ∇_t^E is a connection on E such that $\nabla_0^E = \nabla^E$ and $\nabla_1^E = \widetilde{\nabla}^E$. Moreover,

$$\frac{d\nabla_t^E}{dt} = \widetilde{\nabla}^E - \nabla^E \in \Omega^1(M, \mathrm{End}(E)). \tag{2.1.20}$$

Let R_t^E be the curvature of ∇_t^E.

Let $f'(x)$ be the power series obtained from the derivative of $f(x)$. Then from Proposition 2.1.4 and (2.1.18),

$$\begin{aligned}\frac{d}{dt}\,\mathrm{tr}[f(R_t^E)] &= \mathrm{tr}\left[\frac{dR_t^E}{dt} f'(R_t^E)\right] = \mathrm{tr}\left[\frac{d(\nabla_t^E)^2}{dt} f'(R_t^E)\right] \\ &= \mathrm{tr}\left[\left[\nabla_t^E, \frac{d\nabla_t^E}{dt}\right] f'(R_t^E)\right] = \mathrm{tr}\left[\left[\nabla_t^E, \frac{d\nabla_t^E}{dt} f'(R_t^E)\right]\right] \\ &= d\,\mathrm{tr}\left[\frac{d\nabla_t^E}{dt} f'(R_t^E)\right]. \end{aligned} \tag{2.1.21}$$

By (2.1.21), we have

$$\begin{aligned}\mathrm{tr}\left[f(R^E)\right] - \mathrm{tr}\left[f(\widetilde{R}^E)\right] &= \int_0^1 \frac{d}{dt}\,\mathrm{tr}[f(R_t^E)]dt \\ &= d\left(\int_0^1 \mathrm{tr}\left[\frac{d\nabla_t^E}{dt} f'(R_t^E)\right] dt\right). \end{aligned} \tag{2.1.22}$$

The proof of our theorem is completed. □

Let

$$g(x) = b_0 + b_1 x + \cdots + b_n x^n + \cdots, \quad b_i \in \mathbb{C}, \tag{2.1.23}$$

be a power series in one variable.

Corollary 2.1.6. *(1) The form* $g\left(\operatorname{tr}\left[f(R^E)\right]\right)$ *is closed. That is,*

$$dg\left(\operatorname{tr}\left[f(R^E)\right]\right) = 0. \tag{2.1.24}$$

(2) If $\widetilde{\nabla}^E$ *is another connection on* E *with curvature* $\widetilde{R}^E$*, letting* $\nabla_t^E = (1-t)\nabla^E + t\widetilde{\nabla}^E$*, we have*

$$g\left(\operatorname{tr}\left[f(R^E)\right]\right) - g\left(\operatorname{tr}\left[f(\widetilde{R}^E)\right]\right) = d\left(\int_0^1 g'\left(\operatorname{tr}\left[f(R_t^E)\right]\right) \cdot \operatorname{tr}\left[\frac{d\nabla_t^E}{dt} f'(R_t^E)\right] dt\right). \tag{2.1.25}$$

By Theorem 2.1.5 (1), $g\left(\operatorname{tr}\left[f\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right]\right)$ is a closed differential form which determines a cohomology class $\left[g\left(\operatorname{tr}\left[f\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right]\right)\right]$ in $H^*(M,\mathbb{C})$. While Theorem 2.1.5 (2) says that this class does not depend on the choice of the connection ∇^E.

Definition 2.1.7. (1) The differential form $g\left(\operatorname{tr}\left[f\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right]\right)$ is called the **Characteristic form** of E associated with ∇^E, f and g.

(2) The cohomology class $\left[g\left(\operatorname{tr}\left[f\left(\frac{\sqrt{-1}}{2\pi}R^E\right)\right]\right)\right]$ is called the **Characteristic class** of E associated with f and g.

From (1.3.21), for $R^E \in \Omega^2(M, \operatorname{End}(E))$, we have

$$\det\left(I + \frac{\sqrt{-1}}{2\pi}R^E\right) = \exp\left(\operatorname{tr}\left[\log\left(I + \frac{\sqrt{-1}}{2\pi}R^E\right)\right]\right) \tag{2.1.26}$$

in view of

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}, \quad \exp(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}. \tag{2.1.27}$$

Here I is the identity endomorphism of E.

Definition 2.1.8. The (total) **Chern form**, denoted by $c(E,\nabla^E)$, associated with ∇^E is defined by

$$c(E,\nabla^E) = \det\left(I + \frac{\sqrt{-1}}{2\pi}R^E\right). \tag{2.1.28}$$

We see that $c(E,\nabla^E)$ is a characteristic form in the sense of Definition 2.1.7. The associated characteristic class, denoted by $c(E)$, is called the (total)

Chern class of E. By (2.1.28), we have the decomposition of the (total) Chern form that

$$c(E,\nabla^E)=1+c_1(E,\nabla^E)+\cdots+c_k(E,\nabla^E)+\cdots \tag{2.1.29}$$

with

$$c_i(E,\nabla^E)\in\Omega^{2i}(M). \tag{2.1.30}$$

We call $c_i(E,\nabla^E)$ the i-th **Chern form** associated with ∇^E, and its associated cohomology class, denoted by $c_i(E)$, the i-th **Chern class** of E.

It is easy to see that if E is a trivial bundle, $c(E)=1$.

Especially, by (2.1.26)-(2.1.29), the first Chern form

$$c_1(E,\nabla^E)=\frac{\sqrt{-1}}{2\pi}\operatorname{tr}[R^E]\in\Omega^2(M). \tag{2.1.31}$$

We rewrite (2.1.26) as

$$\log\left(\det\left(I+\frac{\sqrt{-1}}{2\pi}R^E\right)\right)=\operatorname{tr}\left[\log\left(I+\frac{\sqrt{-1}}{2\pi}R^E\right)\right]. \tag{2.1.32}$$

The from the power series expansion of $\log(1+x)$, we can deduce that for any integer $k\geq 0$, $\operatorname{tr}\left[(R^E)^k\right]$ can be written as a linear combination of various products of $c_i(E,\nabla^E)$'s.

Therefore, by Definition 2.1.7, any characteristic form (or characteristic class) could be written as a linear combination of various products of $c_i(E,\nabla^E)$'s (or $c_i(E)$'s). This establish the fundamental importance of the Chern class in the theory of characteristic classes of complex vector bundles.

Proposition 2.1.9. *Let E_1, E_2 be two vector bundles over M endowed with connections ∇^{E_1} and ∇^{E_2} respectively. Let R^{E_1} and R^{E_2} be the corresponding curvatures.*

(1) The curvature of the induced connection on the direct sum $E_1\oplus E_2$ is given by

$$R^{E_1\oplus E_2}=R^{E_1}\oplus R^{E_2}. \tag{2.1.33}$$

(2) On the tensor product $E_1\otimes E_2$, the induced curvature is given by

$$R^{E_1\otimes E_2}=R^{E_1}\otimes 1\oplus 1\otimes R^{E_2}. \tag{2.1.34}$$

(3) Let E^* be the dual of E, we have

$$R^{E^*}=-(R^E)^t. \tag{2.1.35}$$

(4) For a smooth map $f:N\to M$, we have

$$R^{f^*E}=f^*R^E. \tag{2.1.36}$$

Proposition 2.1.10. *Let E, E' be complex vector bundles over M endowed with connections ∇^E, $\nabla^{E'}$ respectively.*

(1) Let $\nabla^{E\oplus E'}$ be the induced connection on $E\oplus E'$,

$$c(E\oplus E', \nabla^{E\oplus E'}) = c(E,\nabla^E)\cdot c(E',\nabla^{E'}). \tag{2.1.37}$$

(2) Let $\nabla^{E\otimes E'}$ be the induced connection on $E\otimes E'$. If E' is a line bundle,

$$c_1(E\otimes E', \nabla^{E\otimes E'}) = c_1(E) + \operatorname{rank}(E)\cdot c_1(E'). \tag{2.1.38}$$

(3) Let ∇^{E^*} be the induced connection on the dual bundle E^*,

$$c_i(E^*, \nabla^{E^*}) = (-1)^i c_i(E,\nabla^E). \tag{2.1.39}$$

(4) Let $f : N \to M$ be a smooth map. Let $f^*\nabla^E$ be the induced connection on f^*E,

$$c_i(f^*E, f^*\nabla^E) = f^* c_i(E,\nabla^E). \tag{2.1.40}$$

(5) If $\operatorname{rank}(E) = k$, *then*

$$c_1(E,\nabla^E) = c_1(\Lambda^k E, \nabla^{\Lambda^k E}). \tag{2.1.41}$$

Proof. Note that (1), (3) and (4) follow directly from Proposition 2.1.9. We only need to prove (2) and (5).

From Proposition 2.1.9 (2), we have

$$\operatorname{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^{E\otimes E'}\right)\right] = \operatorname{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi}R^{E}\right)\right]\cdot(1 + c_1(E',\nabla^{E'})). \tag{2.1.42}$$

Then (2) follows from taking the 2-form part of the two sides of (2.1.39).

Let $\nabla^{\Lambda^k E}$ be the connection on $\Lambda^k E$ induced from ∇^E. Let Γ and $\widetilde{\Gamma}$ be the connection forms of ∇^E and $\nabla^{\Lambda^k E}$. Let $\sigma_1, \cdots, \sigma_k$ be a local basis of sections of E. Then

$$\begin{aligned}\nabla^{\Lambda^k E}(\sigma_1\wedge\cdots\wedge\sigma_k) &= \sum_{i=1}^k \sigma_1\wedge\cdots\wedge\nabla^E\sigma_i\wedge\cdots\wedge\sigma_k \\ &= \sum_{i=1}^k \Gamma_{ii}\sigma_1\wedge\cdots\wedge\sigma_k.\end{aligned} \tag{2.1.43}$$

So we have

$$\widetilde{\Gamma} = \operatorname{tr}[\Gamma]. \tag{2.1.44}$$

Thus

$$\operatorname{tr}[R^E] = \operatorname{tr}[d\Gamma] + \operatorname{tr}[\Gamma \wedge \Gamma] = d\operatorname{tr}[\Gamma] = d\widetilde{\Gamma} = R^{\Lambda^k E}. \tag{2.1.45}$$

Therefore we get (5).

The proof of our proposition is completed. □

Example 2.1.11 (First Chern form of Chern connection)**.** Let M be a complex manifold and E be a holomorphic vector bundle over M with Hermitian metric h^E. Let ∇^E be the Chern connection on (E, h^E). By Theorem 1.2.11, the curvature of the Chern connection is

$$R^E = \bar{\partial}\partial \log(h^E). \tag{2.1.46}$$

Then the first Chern form

$$\begin{aligned} c_1(E, \nabla^E) = \frac{\sqrt{-1}}{2\pi} \operatorname{tr}[R^E] &= -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} \operatorname{tr}\log(h^E) \\ &= -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\det(h^E) \in \Omega^{1,1}(M). \end{aligned} \tag{2.1.47}$$

Example 2.1.12 (γ_n on $\mathbb{CP}^n$)**.** By (1.2.33),

$$c_1(\gamma_n, \nabla^{\gamma_n}) = \frac{\sqrt{-1}}{2\pi} R^{\gamma_n} = -\frac{\sqrt{-1}}{2\pi} \frac{(1+|\theta|^2)d\theta_i \wedge d\bar{\theta}_i - \bar{\theta}_i\theta_j d\theta_i \wedge d\bar{\theta}_j}{(1+|\theta|^2)^2}. \tag{2.1.48}$$

From (1.2.34), we have

$$c_1(\gamma_n, \nabla^{\gamma_n}) = -\frac{1}{2\pi}\omega_{FS}, \tag{2.1.49}$$

where ω_{FS} is the Kähler form associated with the Fubini-Study metric on $\mathbb{CP}^n$.

If $n = 1$,

$$\int_{\mathbb{CP}^1} \omega_{FS} = \int_{\mathbb{C}} \frac{\sqrt{-1}d\theta \wedge d\bar{\theta}}{(1+|\theta|^2)^2} = \int_0^{2\pi}\int_0^{+\infty} \frac{2r}{(1+r^2)^2} dr d\varphi = 2\pi. \tag{2.1.50}$$

So from (2.1.49),

$$\int_{\mathbb{CP}^1} c_1(\gamma_1, \nabla^{\gamma_1}) = -1. \tag{2.1.51}$$

That means the first Chern class of the tautological bundle of $\mathbb{CP}^1$ is equal to -1 in $H^2(\mathbb{CP}^1, \mathbb{Z}) \simeq \mathbb{Z}$.

Definition 2.1.13. Let M be a complex manifold with complex dimension n. Then the holomorphic line bundle

$$K_M := T^{*(n,0)}M \tag{2.1.52}$$

is called the **canonical line bundle** of M.

Definition 2.1.14. Let M be a complex manifold. Let $T^{\mathbb{C}}M$ be the complex tangent bundle of M. We define

$$c_i(M) := c_i(T^{\mathbb{C}}M) = c_i(T^{(1,0)}M) \in H^{2i}(M, \mathbb{C}), \tag{2.1.53}$$

which is called the i-th **Chern class** of M.

From Proposition 2.1.10 (5), we have

$$c_1(M) = c_1(K_M^*) = -c_1(K_M). \tag{2.1.54}$$

If M is a complex manifold, by 1.1.7, $\bar{\partial}^2 = 0$. The following definition is well-defined.

Definition 2.1.15. Let M be a complex manifold. Then the (p,q)-Dolbeault cohomology is the vector space

$$H^{p,q}(M) := \frac{\operatorname{Ker}(\bar{\partial}|_{\Omega^{p,q}(M)})}{\operatorname{Im}(\bar{\partial}|_{\Omega^{p,q-1}(M)})}. \tag{2.1.55}$$

Note that if $\alpha \in \Omega^{p,q}$ is d-closed, it is $\bar{\partial}$-closed. It means that $[\alpha] \in H^{p,q}(M)$. So

- if E is a holomorphic vector bundle over M, $c_i(M) \in H^{i,i}(M)$;
- if (M, ω) is Kähler, $[\omega] \in H^{1,1}(M)$;
- if (M, ω) is Kähler, $[\mathrm{Ric}_\omega] \in H^{1,1}(M)$.

From Proposition 1.3.4, Definition 2.1.14 and (2.1.47), we have the following proposition.

Proposition 2.1.16. *Let (M, ω) be a Kähler manifold. Then the first Chern form of $T^{(1,0)}M$ associated with its Chern connection is*

$$c_1(T^{(1,0)}M, \nabla^{T^{(1,0)}M}) = \frac{\sqrt{-1}}{2\pi} R^{K_M^*} = -\frac{\sqrt{-1}}{2\pi} R^{K_M} = \frac{1}{2\pi} \mathrm{Ric}_\omega. \tag{2.1.56}$$

Here Ric_ω is the Ricci form in Definition 1.3.1. Moreover, for the Chern class,

$$c_1(M) = \left[\frac{1}{2\pi} \mathrm{Ric}_\omega\right] \in H^{1,1}(M). \tag{2.1.57}$$

Theorem 2.1.17 (Calabi, Yau)**.** *Let (M,ω) be a Kähler manifold. For any $\rho \in [2\pi c_1(M)]$, there exists uniquely Kähler form ω' satisfying $[\omega'] = [\omega] \in H^{1,1}(M)$ such that $\mathrm{Ric}_{\omega'} = \rho$. In particular, if $c_1(M) = 0$, there exists Kähler form ω' such that $\mathrm{Ric}_{\omega'} = 0$, i.e., ω' is Ricci-flat or (M,ω') is a Kähler-Einstein manifold with Einstein constant 0.*

From (1.1.12) and (1.1.22), for Kähler form ω, we could calculate that

$$\begin{aligned}\omega^n &= (\sqrt{-1})^n g_{i_1,\bar{j}_1}\cdots g_{i_n,\bar{j}_n} dz^{i_1}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge dz^{i_n}\wedge d\bar{z}^{j_n}\\ &= (\sqrt{-1})^n g_{i_1,\bar{j}_1}\cdots g_{i_n,\bar{j}_n}\delta_{i_1,\cdots,i_n}\delta_{j_1,\cdots,j_n} dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^n\\ &= (\sqrt{-1})^n g_{1,\bar{j}_{i_1}}\cdots g_{n,\bar{j}_{i_n}}\delta_{i_1,\cdots,i_n}\delta^{j_{i_1},\cdots,j_{i_n}}_{j_1,\cdots,j_n}\delta_{j_{i_1},\cdots,j_{i_n}} dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^n\\ &= n!(\sqrt{-1})^n g_{1,\bar{j}_1}\cdots g_{n,\bar{j}_n}\delta_{j_1,\cdots,j_n} dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^n\\ &= n!(\sqrt{-1})^n \det(g_{i\bar{j}}) dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^n\\ &= 2^n n!\det(g_{i\bar{j}}) dx^1\wedge dy^1\wedge\cdots\wedge dx^n\wedge dy^n. \qquad (2.1.58)\end{aligned}$$

It means that ω^n is a volume form of M.

Definition 2.1.18. (1) A real (1,1)-form φ on a complex manifold M is called positive (resp. negative) if the symmetric tensor $\varphi(\cdot, J\cdot)$ is positive (resp. negative) definite. If $\varphi > 0$ (resp. $\varphi < 0$), as in (2.1.58), we have $\int_M \varphi^n > 0$ (resp. $\int_M \varphi^n < 0$).

(2)A cohomology class in $H^{1,1}(M)\cap H^2(M,\mathbb{R})$ is called positive (resp. negative) if it can be represented by a positive (resp. negative) (1,1)-form. (For the well-definedness of this definition, we need to figure out that if φ and φ' are two representatives of the cohomology, it is not possible that $\varphi > 0$ and $\varphi' < 0$. If not, we have $\int_M \varphi^n > 0$, $\int_M (\varphi')^n < 0$ and $(\varphi')^n - \varphi^n$ is d-exact. But by Stokes' formula, it is not possible.)

(3) A holomorphic line bundle L over a compact complex manifold is called positive (resp. negative) if there exists a Hermitian structure on L with Chern connection ∇^L and curvature $R^L = (\nabla^L)^2$ such that $\sqrt{-1}R^L$ is a positive (resp. negative) (1,1)-form.

From (2.1.31), it is easy to see that

$$L > 0 \Leftrightarrow c_1(L) > 0, \quad L < 0 \Leftrightarrow c_1(L) < 0. \qquad (2.1.59)$$

Proposition 2.1.19. *If there exists a complex line bundle L over M such that $c_1(L) > 0$, then M is Kähler. Note that if $c_1(L) < 0$, $c_1(L^*) > 0$.*

Proof. From Definition 2.1.18, there exists a positive $(1,1)$-form φ such that $[\varphi] = c_1(L)$. Since $\varphi(\cdot, J\cdot)$ is positive definite, we take $g(\cdot,\cdot) := \varphi(\cdot, J\cdot)$ as the metric on M. Then φ is a closed Kähler form.

The proof of our proposition is completed. □

From (1.3.14) and (2.1.18) if (M, ω) is a Kähler-Einstein manifold with Einstein constant k, then

$$c_1(M) = k \cdot \left[\frac{1}{2\pi}\omega\right]. \tag{2.1.60}$$

Since $\omega > 0$, if $k > 0$ (resp. < 0), $c_1(M) > 0$ (resp. < 0).

Theorem 2.1.20 (Aubin, Yau)**.** *Let (M, ω) be a compact Kähler manifold. If $c_1(M) < 0$, there exists a unique Kähler-Einstein metric on M up to scalar factors.*

Comparing with Theorem 1.3.16, a recent result says that the negative holomorphic sectional curvature implies that the first Chern class is negative.

Theorem 2.1.21 (Wu-Yau, Tosatti-Yang, Diverio-Trapani)**.** *Let (M, ω) be a compact Kähler manifold with negative holomorphic sectional curvature. Then $c_1(M) < 0$.*

Definition 2.1.22. If M is a compact complex manifold with $c_1(M) > 0$, then it is called the **Fano manifold**.

By Proposition 2.1.19, the Fano manifold is Kähler.

Theorem 2.1.23 (Chen-Donaldson-Sun, Tian)**.** *Let (M, ω) be a compact Kähler manifold. If $c_1(M) > 0$, there exists a Kähler-Einstein metric on M if and only if M is K-stable.*

Note that in local coordinates,

$$c_1(T^{(1,0)}M, \nabla^{T^{(1,0)}M}) = -\frac{\sqrt{-1}}{2\pi}\frac{\partial^2 \log\det(g)}{\partial z_i \partial \bar{z}_j} dz^i \wedge d\bar{z}^j. \tag{2.1.61}$$

Example 2.1.24. We now compute the Chern class of $\mathbb{CP}^n$.

Let E be the orthogonal complement of γ_n using the standard Hermitian metric on $\mathbb{C}^{n+1}$, such that $\gamma_n \oplus E$ is a trivial complex vector bundle over $\mathbb{CP}^n$ with complex rank $n+1$. From the theory of vector bundles (cf. eg. Milnor "Characteristic class" Theorem 14.10), we could obtain that

$$T^{\mathbb{C}}\mathbb{CP}^n \simeq \mathrm{Hom}_{\mathbb{C}}(\gamma_n, E). \tag{2.1.62}$$

Observe that for complx line bundle γ_n, $\mathrm{Hom}_{\mathbb{C}}(\gamma_n, \gamma_n) \simeq \gamma_n^* \otimes \gamma_n$ is a trivial line bundle. (In fact, this result holds for all line bundles.) By adding the trivial line bundle on two sides of (2.1.62), we have

$$T^{\mathbb{C}}\mathbb{CP}^n \oplus \mathcal{E}^1 \simeq \mathrm{Hom}_{\mathbb{C}}(\gamma_n, \mathcal{E}^{n+1}). \tag{2.1.63}$$

Here $\mathcal{E}^k$ denotes the trivial complex vector bundle with rank k. Clearly the right hand side of (2.1.63) can be identified with the Whitney sum of $n+1$ copies of the dual bundle $\mathrm{Hom}_{\mathbb{C}}(\gamma_n, \mathcal{E}^1) \simeq \gamma_n^*$. Thus by Proposition 2.1.10 (1), (3), we have

$$c(\mathbb{CP}^n) = c(T^{\mathbb{C}}\mathbb{CP}^n \oplus \mathcal{E}^1) = c(\gamma_n^*)^{n+1} = (1 - c_1(\gamma_n))^{n+1}. \tag{2.1.64}$$

In particular,

$$c_1(\mathbb{CP}^n) = -(n+1)c_1(\gamma_n) = (n+1)\left[\frac{1}{2\pi}\omega^{FS}\right]. \tag{2.1.65}$$

From Proposition 2.1.16, we get the result again that $\mathbb{CP}^n$ is a Kähler-Einstein manifold with Einstein constant $n+1$.

From (2.1.64), $c_2(\mathbb{CP}^n) = \frac{n(n+1)}{2}c_1(\gamma_n)$. Combining with (2.1.65), we have

$$nc_1(\mathbb{CP}^n)^2 = 2(n+1)c_2(\mathbb{CP}^n). \tag{2.1.66}$$

For complex projective space, by (2.1.58),

$$\begin{aligned} \det\left(g^{FS}_{i\bar{j}}\right) &= \frac{\det\left((1+|\theta|^2)I - \bar{\theta}^t\theta\right)}{(1+|\theta|^2)^{2n}} \\ &= \frac{(1+|\theta|^2)^n \det(1 - |\theta|^2(1+|\theta|^2)^{-1})}{(1+|\theta|^2)^{2n}} = \frac{1}{(1+|\theta|^2)^{n+1}}. \end{aligned} \tag{2.1.67}$$

Here we use the identity of determinants:

$$\begin{aligned} \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \det(A)\det(D - CA^{-1}B) \\ &= \det(D)\det(A - BD^{-1}C), \end{aligned} \tag{2.1.68}$$

for A, D invertible.

we could calculate that

$$\begin{aligned} &\int_{\mathbb{CP}^n} \left(\omega^{FS}\right)^n = 2^n n! \int_{U_0} \frac{1}{(1+|x|^2)^{n+1}} dx^1 \wedge \cdots \wedge dx^{2n} \\ &= 2^n\pi^n n \int_0^{+\infty} \frac{2r^{2n-1}}{(1+r^2)^{n+1}} dr = 2^n\pi^n n \cdot n^{-1}\left(\frac{r^2}{1+r^2}\right)\Bigg|_0^{\infty} = (2\pi)^n. \end{aligned} \tag{2.1.69}$$

Here we use the formula

$$\int_{\mathbb{R}^n} f(|x|)dx^1 \wedge \cdots \wedge dx^{2n} = \frac{2\pi^n}{(n-1)!}\int_0^{+\infty} r^{2n-1}f(r)dr. \tag{2.1.70}$$

Therefore, we have

$$\int_{\mathbb{CP}^n} c_1(\mathbb{CP}^n)^n = (n+1)^n. \tag{2.1.71}$$

Theorem 2.1.25 (Miyaoka-Yau Inequality)**.** *Let X^n be a compact Kähler manifold.*

(1) If M is Kähler-Einstein with $k > 0$, then

$$n \int_M c_1(M)^n \leq 2(n+1) \int_M c_1(M)^{n-2} c_2(M), \tag{2.1.72}$$

with equality if and only if $M = \mathbb{CP}^n$.

(2) If M is Kähler-Einstein with $k < 0$, then

$$n(-1)^{n-2} \int_M c_1(M)^n \leq 2(n+1)(-1)^{n-2} \int_M c_1(M)^{n-2} c_2(M), \tag{2.1.73}$$

with equality if and only if $M = \mathbb{H}_c^n / \Gamma$.

Theorem 2.1.26 (Fujita '18)**.** *If M^n is a Fano manifold with Kähler-Einstein metric, then*

$$\int_M c_1(M)^n \leq (n+1)^n, \tag{2.1.74}$$

with equality if and only if M is biholomorphic to $\mathbb{CP}^n$.

Remark 2.1.27. For the line bundles, the first Chern class is a complete invariant. It means that for any element in $H^2(M, \mathbb{Z})$, there exists a line bundle such that this element is the first Chern class of this bundle and if two line bundles are not isomorphic, then the first Chern classes of them are not equal. This is not right for vector bundles with higher rank.

We now list some other common characteristic classes here.

- The **Chern character form** associated with ∇^E is defined by

$$\mathrm{ch}(E, \nabla^E) = \mathrm{tr}\left[\exp\left(\frac{\sqrt{-1}}{2\pi} R^E\right)\right] \in \Omega^{\mathrm{even}}(M). \tag{2.1.75}$$

  The associated cohomology class, denoted by $\mathrm{ch}(E)$, is called the **Chern character** of E. For complex vector bundles E_1, E_2,

$$\mathrm{ch}(E_1 \oplus E_2) = \mathrm{ch}(E_1) + \mathrm{ch}(E_2), \tag{2.1.76}$$

$$\mathrm{ch}(E_1 \otimes E_2) = \mathrm{ch}(E_1)\, \mathrm{ch}(E_2). \tag{2.1.77}$$

  The Chern character is a polynomial with respect to the Chern classes:

$$\mathrm{ch} = r + c_1 + \frac{1}{2}(-2c_2 + c_1^2) + \frac{1}{6}(c_1^3 - 3c_1 c_2 + 3c_3) + \cdots . \tag{2.1.78}$$

  Here $r = \operatorname{rank} E$.

- The **Todd form** associated with ∇^E is defined by

$$\mathrm{Td}(E, \nabla^E) = \det \left(\frac{\frac{\sqrt{-1}}{2\pi} R^E}{1 - \exp\left(-\frac{\sqrt{-1}}{2\pi} R^E\right)} \right) \in \Omega^{\mathrm{even}}(M). \tag{2.1.79}$$

The associated cohomology class, denoted by $\mathrm{Td}(E)$, is called the **Todd class** of E. The Todd class is a polynomial with respect to the Chern classes:

$$\mathrm{Td} = \frac{1}{2} c_1 + \frac{1}{12}(c_2 + c_1^2) + \frac{1}{24} c_1 c_2 + \cdots . \tag{2.1.80}$$

Recall that in (1.2.17), for holomorphic vector bundle E, $\bar{\partial}^E : \Omega^{p,q}(M, E) \to \Omega^{p,q+1}(M, E)$ is well-defined and $\left(\bar{\partial}^E\right)^2 = 0$.

Definition 2.1.28. Let M be a complex manifold and E be a holomorphic vector bundle. Then the Dolbeault cohomology $H^q(M, E)$ is the vector space

$$H^q(M, E) := \frac{\mathrm{Ker}(\bar{\partial}^E|_{\Omega^{0,q}(M,E)})}{\mathrm{Im}(\bar{\partial}^E|_{\Omega^{0,q-1}(M,E)})}. \tag{2.1.81}$$

We also denote by

$$H^{p,q}(M, E) := H^q(M, \Lambda^p T^{*(1,0)} M \otimes E). \tag{2.1.82}$$

Theorem 2.1.29 (Hirzebruch-Riemann-Roch Theorem)**.** *Let M be a complex manifold and E be a holomorphic vector bundle. Then*

$$\sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} H^i(M, E) = \int_M \mathrm{Td}(T^{1,0} M) \,\mathrm{ch}(E). \tag{2.1.83}$$

Remark 2.1.30 (Characteristic class for real bundle)**.** Let now E be a real vector bundle over M, and ∇^E be a connection on E. Let R^E be the curvature of E. Proceeding in exactly the same way as (2.1.6)-Definition 2.1.7 for real vector bundles with connections, we could also get Chern-Weil theory for real vector bundles. In the following examples, we assume that E is a real bundle.

- The **Pontrjagin form** associated with ∇^E is defined by

$$p(E, \nabla^E) = \det \left(\left(I - \left(\frac{R^E}{2\pi} \right)^2 \right)^{1/2} \right) \in \Omega^{4k}(M). \tag{2.1.84}$$

The associated cohomology class, denoted by $p(E)$, is called the **Pontrjagin class** of E. As the Chern form, $p(E, \nabla^E)$ admits a decomposition

$$p(E, \nabla^E) = 1 + p_1(E, \nabla^E) + \cdots + p_k(E, \nabla^E) + \cdots \tag{2.1.85}$$

with $p_i(E, \nabla^E)\Omega^{4i}(M)$. We call $p_i(E, \nabla^E)$ the i-th Pontrjagin form associated with ∇^E and the associated class $p_i(E)$ the i-th Pontrjagin class of E. For $i \geq 0$,

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C}). \tag{2.1.86}$$

- The **Hirzebruch's L-form** associated with ∇^E is defined by

$$L(E, \nabla^E) = \det\left(\left(\frac{\frac{\sqrt{-1}}{2\pi}R^E}{\tanh\left(\frac{\sqrt{-1}}{2\pi}R^E\right)}\right)^{1/2}\right) \in \Omega^{4k}(M). \tag{2.1.87}$$

The associated cohomology class, denoted by $L(E)$, is called the **L-class** of E. The L-class is a polynomial with respect to the Pontrjagin classes:

$$L = \frac{1}{3}p_1 + \frac{1}{45}(7p_2 - p_1^2) + \frac{1}{3^3 \cdot 5 \cdot 7}(62p_3 - 13p_1p_2 + 2p_1^3) + \cdots . \tag{2.1.88}$$

- The **Â-form** associated with ∇^E is defined by

$$\hat{A}(E, \nabla^E) = \det\left(\left(\frac{\frac{\sqrt{-1}}{4\pi}R^E}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^E\right)}\right)^{1/2}\right) \in \Omega^{4k}(M). \tag{2.1.89}$$

The associated cohomology class, denoted by $\hat{A}(E)$, is called the **Â-class** of E. The Â-class is a polynomial with respect to the Pontrjagin classes:

$$\hat{A} = -\frac{1}{24}p_1 + \frac{1}{2^7 \cdot 3^2 \cdot 5}(-4p_2 + 7p_1^2) - \frac{1}{2^{10} \cdot 3^3 \cdot 5 \cdot 7}(16p_3 - 44p_1p_2 + 31p_1^3) + \cdots . \tag{2.1.90}$$

If E is oriented,

$$\mathrm{Td}(E \otimes \mathbb{C}) = \hat{A}(E)^2. \tag{2.1.91}$$